Math 010 - Exam 2 Study Guide

This study guide is designed to help you prepare specifically for Exam 2. Every item below corresponds directly to skills or ideas that appear on the exam. Use the practice worksheet at the end to test your readiness.

1. Linear Transformations: Definitions and Computation

What you should be able to do:
- Identify the domain and codomain of a transformation $T: R^n \rightarrow R^m$ from its formula.
- Build the standard matrix $[T]$ by computing T applied to each standard basis vector.
- Evaluate T at a specific vector by direct substitution or by computing $[T]\mathbf{x}$.

You should understand:
- Why the columns of $[T]$ are exactly the images of the standard basis vectors.
- The difference between the domain (input space) and codomain (output space) and how the dimensions of $[T]$ reflect both.

2. Testing Whether a Transformation is Linear

What you should be able to do:
- Check both linearity conditions: additivity $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + \mathrm{T}(\mathbf{v})$ and homogeneity $T(k\mathbf{u}) = kT(\mathbf{u})$.
- Write a complete justification either confirming linearity or disproving it.

You should understand:
- That disproving linearity requires only one counterexample, while proving it requires a general argument.

3. Geometric Transformations and Composition

What you should be able to do:
- Write down the standard matrices for common geometric operators on R^2: rotations (by any angle), reflections (about coordinate axes or the line $y = x$), and orthogonal projections (onto coordinate axes).
- Form the standard matrix of a composition of two transformations by multiplying their matrices in the correct order.
- Apply the resulting composition matrix to a given vector.

You should understand:
- Why order matters: the transformation applied *first* corresponds to the matrix on the *right* in the product.
- How to derive standard matrices if you don't remember them.

4. Applications

What you should be able to do:

- Translate the condition that a polynomial $a_0 + a_1 x + \cdots + a_n x^n$ passes through a given point into a linear equation in the unknowns $a_0, a_1, \ldots, a_n$.
- Translate a chemical equation into a system of equations.
- Interpret the diagram of an electrical circuit in terms of linear equations.
- Assemble the augmented matrix from given data points
- Stop reducing as soon as back-substitution becomes straightforward—you do not need full RREF

You should understand:

- Why n distinct points determine a unique degree-$n-1$ polynomial (the system has a unique solution).
- Why a system associated with a chemical equation has infinitely many solutions.

5. Determinants and Finding When $\det(A) = 0$

What you should be able to do:

- Compute the determinant of a 3×3 matrix by cofactor expansion (choose a row or column with zeros to simplify work, if possible)
- Set up and solve the equation $\det(A) = 0$ when A contains an unknown λ.
- Factor the resulting polynomial to find all values of λ.

You should understand:

- How to take advantage of zeros in a row or column to reduce the number of cofactor terms you must compute.
- The algebraic significance of $\det(A) = 0$ (the matrix is singular/not invertible).

6. Properties of Determinants

What you should be able to do:

- Apply the key determinant rules to compute determinants of scalar multiples, inverses, transposes, powers, and products without re-computing from scratch.
- Know that for an $n \times n$ matrix: $\det(kA) = k^n \det(A)$, $\det(A^{-1}) = \frac{1}{\det(A)}$, $\det(A^T) = \det(A)$, and $\det(AB) = \det(A)\det(B)$.

You should understand:

- Why each property holds conceptually (e.g., scaling every row scales the determinant once per row).
- How to combine these rules for composite expressions like $\det\big((kA^{-1})\big)$ or $\det(A^T A)$.

7. Linear Combinations of Vectors

What you should be able to do:

- Translate the vector equation $a\mathbf{u} + b\mathbf{v} = \mathbf{w}$ into an augmented matrix.
- Solve the system by row reduction and state the values of the scalars or conclude that no such scalars exist.

You should understand:

- Why this is structurally the same as solving $A\mathbf{x} = \mathbf{b}$, with columns of A being the given vectors.

8. Vector Operations: Norm, Dot Product, and Angles

What you should be able to do:

- Compute linear combinations of vectors and find their norms.
- Evaluate the dot product of two vectors.
- Determine whether two vectors are orthogonal and justify your conclusion.
- Find the cosine of the angle between two vectors using the dot product formula.

You should understand:

- The relationship $\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$ and when it implies orthogonality ($\theta = 90°$).
- Why $\|c\mathbf{u} + d\mathbf{v}\|$ requires you to first compute the vector $c\mathbf{u} + d\mathbf{v}$, and then take its norm, not combine norms directly.

9. Vector Operations and Orthogonal Decomposition

What you should be able to do:

- Compute the vector projection of $\mathbf{u}$ onto $\mathbf{a}$: $\mathrm{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}$.
- Find the component of $\mathbf{u}$ orthogonal to $\mathbf{a}$: $\mathbf{u} - \mathrm{proj}_{\mathbf{a}}\mathbf{u}$.

You should understand:

- Why these two components always form an orthogonal decomposition.
- The difference between the scalar projection (a number) and the vector projection (a vector).

10. Conceptual Proofs

What you should be able to do:

- Write component-based proofs of vector and dot-product algebraic identities (e.g., properties of scalar multiplication with the dot product, distributive laws).
- Construct if-and-only-if proofs involving linear transformations and their matrix representations.

You should understand:

- How to use components forms of vectors to prove vector algebraic identities.
- The precise relationship between $T_A(\mathbf{x}) = A\mathbf{x}$ and $T_B(\mathbf{x}) = B\mathbf{x}$**:** composition of transformations corresponds to multiplication of their matrices.

Math 010: Linear Algebra Practice Worksheet

1. Linear Transformations

Let T be defined by $T(x_1, x_2, x_3) = (3x_1 + x_2 - 2x_3, 2x_1 - x_2 + x_3)$.

a. Find the standard matrix $[T]$.
b. State the domain and codomain of $[T]$.
c. Compute $T(2, -1, 3)$ using $[T]$.

2. Testing Linearity

Determine whether $T: R^2 \to R^2$ defined by $T(x_1, x_2) = (x_1 + x_2, x_1 x_2)$ is a linear transformation. Use the linearity conditions and provide a complete justification.

3. Composition of Geometric Transformations

a. Find the standard matrix in R^2 for: a counterclockwise rotation by 30°, followed by a reflection about the x-axis.
b. Compute the image of $(4, -2)$ under this composition.

4. Polynomial Curve Fitting

Find the quadratic polynomial $p(x) = a_0 + a_1 x + a_2 x^2$ passing through the points $(-1, 6)$, $(0, 1)$, and $(2, 7)$ by setting up and row reducing an augmented matrix.

5. Determinants with an Unknown

Find all values of λ for which $\det(A) = 0$, where $A = \begin{bmatrix} \lambda - 2 & 1 & 0 \\ 0 & \lambda - 3 & 0 \\ 2 & 1 & \lambda - 1 \end{bmatrix}$.

(*Hint: Choose your expansion row or column wisely.*)

6. Determinant Properties

Suppose A is a 4×4 matrix with $\det(A) = 5$. Compute:

a. $\det(A^{-1})$

b. $\det(2A)$

c. $\det(A^T A)$

d. $\det(-A)$

7. Linear Combinations

Find scalars a and b such that $a\mathbf{u} + b\mathbf{v} = \mathbf{w}$, where $\mathbf{u} = (1, 2, -1), \mathbf{v} = (3, 1, 2), \mathbf{w} = (7, 5, 0)$ or show that no such scalars exist.

8. Dot Product and Norms

Let $\mathbf{u} = (1, 2, -2)$ and $\mathbf{v} = (3, 0, 1)$. Compute:

a. $\|\mathbf{u} - 2\mathbf{v}\|$

b. $\mathbf{u} \cdot \mathbf{v}$

c. Are $\mathbf{u}$ and $\mathbf{v}$ orthogonal? Justify your answer.

d. Find the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$.

9. Vector Projection

Let $\mathbf{u} = (2, 1, 6)$ and $\mathbf{a} = (1, 2, 2)$.

a. Find $\text{proj}_{\mathbf{a}}\mathbf{u}$.

b. Find the component of $\mathbf{u}$ orthogonal to $\mathbf{a}$ and verify it is orthogonal to $\mathbf{a}$.

c. Verify that your two components in (a) and (b) sum to $\mathbf{u}$.

10. Proof Practice

Proofs will be taken from assigned homework problems.